1. The Lorentz Force on a charged particle is

$$\mathbf{F} = qe(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

The magnetic part of this provides the centripetal force causing a particle to move in a circle so in the non-relativistic case:

$$\begin{aligned} \frac{mv^2}{R} &= eqvB \\ \Rightarrow p &= eqBR \end{aligned}$$

For relativistic particles $m \to \gamma m$ and the equation still holds. This is still in SI units so the momentum is in $kg\,m\,s^{-1}$. We want it in GeV/c:

$$\begin{aligned} 1\,\mathrm{GeV}/c &= \frac{10^9 \times |e| J}{c} \\ &= \frac{10^9 \times |e|\,kg\,m^2\,s^{-2}}{3 \times 10^8\,m\,s^{-1}} \\ &= \frac{|e|\,kg\,m\,s^{-1}}{0.3} \\ \Rightarrow 1\,kg\,m\,s^{-1} &= \frac{0.3}{|e|}\,\mathrm{GeV}/c \\ \Rightarrow eqBR\,kg\,m\,s^{-1} &= \frac{0.3eqBR}{|e|}\,\mathrm{GeV}/c \\ \Rightarrow p &= 0.3qBR\,\mathrm{GeV}/c \end{aligned}$$

2. (a) In a cyclotron $p\,(\mathrm{GeV}/c) = 0.3qBR$. Setting $p = 104\,\mathrm{GeV}/c, q = 1, R = 27000/2\pi = 4.3 \times 10^3\,m$ gives:

$$B = \frac{p}{0.3qR} = \frac{104}{0.3 \times 4300} = 0.081\,T$$

(b) For LHC, $p = 4000\,\mathrm{GeV}/c$ and so $B = 3.1\,T$.

(c) The energy is related to the speed through the relativistic equation $E = \gamma mc^2$. Using reduced units by dropping the c^2, this is $E = \gamma m$. Setting the proton mass $m_p \approx 1\,\mathrm{GeV}$ gives $\gamma = 4\,\mathrm{TeV}/1\,\mathrm{GeV} = 4000$. Therefore:

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-\beta^2}} \\ \Rightarrow \beta &= \sqrt{\frac{\gamma^2 - 1}{\gamma^2}} \\ \Rightarrow \beta &= 0.999999969c \end{aligned}$$

(d) The power loss in Watts (energy lost per second) is given by:

$$P = \frac{1}{6\pi\epsilon_0}\frac{e^2 v^4 \gamma^4}{c^3 R^2}$$

To turn this into energy lost per turn we must multiply by the time for one revolution $= 2\pi R/v$. Working in SI units, and setting $v \approx c$ and $\gamma = E/m = 4000$:

$$\begin{aligned}
\Delta E\,(\text{Energy lost per turn}) &= \frac{e^2\gamma^4}{3\epsilon_0 R} \\
&= \frac{(1.6\times 10^{-19})^2(4000)^4}{3\times 8.9\times 10^{-12}\times 4.3\times 10^3} \\
&= 5.71\times 10^{-17}\,\text{J/turn} \\
&= 360\,\text{eV/turn}
\end{aligned}$$

(e) The number of revolutions per second is $= v/2\pi R = 1.1\times 10^4$. Therefore the energy lost per proton per second $= 0.36$ keV/turn $\times 1.1\times 10^4$ turns $= 4.0\,\text{MeV}/s$. Therefore energy lost for all protons in all bunches is $= 3.9\times 10^6\,\text{eV}\times 1380\times 2.0\times 10^{11}\times 1.6\times 10^{-19}\,J/\,\text{eV} = 175$ Watts. This is the energy that has to be replaced each second.

3. The length of the accelerator is given by:

$$\begin{aligned}
l &= \frac{(\text{Final Energy})}{(\text{Accelerating Field per metre})} \\
&= \frac{500\,\text{GeV}}{10\,MV/m} = 50\,km
\end{aligned}$$

4. (a) Using 4-momenta notation, $p = (E, \bar{p}) = (E, p_x, p_y, p_z)$, and Einstein's equation $E^2 = m_o^2 + p^2$ in natural units, the initial 4-momenta are (we could equally have had the particles travelling along the x or y direction):

$$\begin{aligned}
p_1 &= (E_1, 0, 0, p) = (\sqrt{p^2 + m_p^2}), 0, 0, p) \qquad \text{Beam} \\
p_2 &= (E_2, 0, 0, 0) = (m_p^2, 0, 0, 0) \qquad \text{Target} \\
\Rightarrow s &= (p_1 + p_2)^2 \\
&= (E_1 + E_2)^2 - p^2 \\
&= (E_1 + m_p)^2 - p^2 \\
&= (500.001 + 1)^2 - 500^2 \\
\Rightarrow \sqrt{s} &\approx 32\,\text{GeV}
\end{aligned}$$

(b) The 4-momenta in a collider would be:

$$\begin{aligned}
p_1 &= (E_{beam}, 0, 0, p) \\
p_2 &= (E_{beam}, 0, 0, -p)
\end{aligned}$$

$$\text{Therefore}\; s = (p_1 + p_2)^2 = 4E_{beam}^2$$

So to achieve $\sqrt{s} = 32\,\text{GeV}$, the beam energy only needs to be 16 GeV.

5. (a) Current is total charge per unit time. Each proton and anti-proton goes round the ring in time $\Delta t = 2\pi R/c \approx 90\,\mu$s. Therefore the current is:

$$I = \frac{(N_p + N_{\bar{p}}) \times |e|}{\Delta t} = \frac{2 \times 1380 \times 2.0 \times 10^{11} \times 1.6 \times 10^{-19}}{90 \times 10^{-6}} = 1.0\,\text{Amps}$$

The actual average current is ~ 0.6 Amps as the number of particles in the bunches decreases with time (as does the luminosity).

(b) The total energy is $4\,\text{TeV} \times 1380 \times 2 \times 10^{11} = 3.93 \times 10^{27}\,\text{eV} = 177\,MJ$.

(c) Luminosity $\mathcal{L} = f n_1 n_2 / A$ with $f = 1380/\Delta t$. So:

$$Area = \frac{f n_1 n_2}{\mathcal{L}} = \frac{1380 \times 2.0 \times 10^{11} \times 2.0 \times 10^{11}}{90 \times 10^{-6}\,s \times 2 \times 10^{33}\,\text{cm}^{-2}\,s^{-1}} = 3.1 \times 10^{-4}\,\text{cm}^2$$

Since the density of particles in the beam is not uniform the "luminous" region where most of the collisions is much smaller than this.

(d) One month of running is 30 days $\times$ 23 hrs/day $= 2.5 \times 10^6$ s. Therefore, the integrated luminosity:

$$\begin{aligned} \int_0^{2.5\times10^6} \mathcal{L} dt &= 2.5 \times 10^6 \times 2 \times 10^{33}\,\text{cm}^{-2} \\ &= 0.5 \times 10^{40}\,\text{cm}^{-2} \\ &= 0.5 \times 10^{16}\,barn^{-1} \\ &= 5\,\text{fb}^{-1} \end{aligned}$$

(e) The number of Higgs seen is the integrated luminosity $\mathcal{L}$ times the cross-section σ times the efficiency ϵ:

$$N = \mathcal{L} \times \sigma \times \epsilon = 5\,\text{fb}^{-1} \times 2\,\text{pb} \times 0.02 = 100$$