SYMMETRIES & CONSERVATIONS LAWS

Homework

If you have problems, do not hesitate to contact me:

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Lecture 1

Q 1.1) By considering the first few terms of the expansions, prove that
\[ \exp(A) \exp(B) \neq \exp(A + B) \]
if \(A\) and \(B\) do not commute.
However, show that the equality holds if \(A\) and \(B\) do commute (prove in general, to all orders).

Q 1.2) By expanding the exponential, find an expression for \(\exp(i\alpha A)\) where
\[
A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

Q 1.3) If \([A,B]=B\), find an expression for \(\exp(i\alpha A)B \exp(-i\alpha A)\) (consider to all orders).
Q 2.1) Consider which of the following are groups:

- Integers under Addition
- Integers under Subtraction
- Integers under Multiplication
- Reals under Multiplication

Any violations of the requirements for a group mean that the set and operation do **not** form a group.

Q 2.2) Demonstrate that there is only one group combination table for 3 distinct objects, i.e. all groups for 3 objects have the same form (are isomorphic) to $Z_3$. Do this by considering all the combinations for 3 distinct objects {e, a, b}.

Q 2.3) Show that the set of Lorentz Transformations:

$$g(\beta) = \begin{cases} 
  x' = \gamma(x - \beta t) \\
  t' = \gamma(t - \beta x) \\
  \gamma = 1/\sqrt{1-\beta^2}
\end{cases}$$

form an Abelian Lie group under the operation “follows”. 

*Hint*: start by combining two boosts: $g(\beta_2)\ g(\beta_1)$ and showing that these correspond to a third boost. Do this in Euclidean space. Easiest to employ matrix notation.

Q 2.4) Show that U(n) and SU(n) are groups.
Q 3.1) Consider rotations in 3D about the x-, y- and z-axes – SO(3).
Identify generators appropriate to
   a) Scalar wavefunctions $\psi(x)$ – we have done this in the Lectures; you can just write down the QM operators (do not write lots of blah, just write down operators)
   b) Real vectors in 3D space – consider infinitesimal rotation matrices; the generators will be 3×3 matrices (give the rotation matrices, consider small angles and identify generators)
In both cases, find the structure constants. (Don’t work out every single possibility, but appeal to symmetry.)

Q 3.2) For the generators $\{L_x,L_y,L_z\}$ in question 3.1, part (b), find the simultaneous eigenvectors of $L^2$ and $L_z$ (i.e. the one set of vectors which are eigenvectors of both operators).

Q 3.3) Find the adjoint matrices for the generators in question 3.1, part (b).
In this case, it is obvious that they satisfy the Lie algebra.

Q 3.4) Verify the form of $R_y(\theta)$ for spin-1 given in Lecture 3.
Lecture 4

Q 4.1) Using the Young Tableaux rules, write down the multiplicity for \( p \) particles of SU(n) in a totally symmetric state, namely a row of \( p \) boxes:

\[
\begin{array}{ccccccc}
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

Now consider examples of how the corresponding states could be labelled by supplying quantum numbers \{1,2,...,n\}.

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 & \\
1 & 1 & 1 & 1 & 1 & 2 & 2 \\
\end{array}
\]

Etc

By considering all the possible configurations, verify the multiplicity. Do this in general, not for a specific example.

*Hint:* This is so trivial, that it requires no algebra, but you have to spot the trick!

The trick is to consider the number of ways of listing \( p \) boxes with \((n-1)\) transitions of state label.

Q 4.2) Using the Young Tableaux rules, verify that the multiplicity of a general multiplet in SU(2) is \((a+1)\) and in SU(3) is \(\frac{1}{2} (a+1)(b+1)(a+b+2)\).
Q 5.1) Considering only flavour, find the ratio of matrix elements for $\pi^0 \rightarrow \gamma\gamma$ and $\eta \rightarrow \gamma\gamma$. Do this for a general case of mixing angle $\theta_p$, and then choose $\theta_p$ such that the $\eta$ has no strange-quark content. Is the $-\ve$ sign in the $\pi^0$ wavefunction meaningful?

*How to proceed:*
Label the scattering operator $S$ and the meson state $|M\rangle$.
What you need is $\langle \gamma\gamma | S | M \rangle = \sum_q \langle \gamma\gamma | S | q\bar{q} \rangle \langle q\bar{q} | M \rangle$

\[
\begin{align*}
\bar{q} & \quad iQ_q & \quad \gamma \\
q & \quad iQ_q & \quad \gamma
\end{align*}
\]

$\langle \gamma\gamma | S | q\bar{q} \rangle \sim Q_q^2$

And

$|\eta\rangle = \cos \theta_p |\eta_8\rangle + \sin \theta_p |\eta_1\rangle$

$|\pi^0\rangle = \frac{1}{\sqrt{2}} (u\bar{u} - d\bar{d})$ etc.