# LECTURE 4 – SU(3)

## Contents

- Gell-Mann Matrices
- QCD
- Quark Flavour SU(3)
- Multiparticle States

## Messages

- Group Theory provides a description of the exchange bosons (gluons) of **QCD** and allows the interactions between **coloured quarks** to be calculated.
- We see how to create **multiplets**, labelled by their weights (quantum numbers), and calculate their multiplicities.

# Gell-Mann Matrices [7.1]

**SU(3)** corresponds to special unitary transformation on complex 3D vectors.

The natural representation is that of 3×3 matrices acting on complex 3D vectors.

There are $3^2-1$ parameters, hence 8 generators: $\{X_1, X_2, \ldots X_8\}$.
The generators are traceless and Hermitian.

The **generators** are derived from the Gell-Mann matrices: $X_i = ½ \lambda_i$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix}, \lambda_8 = \tfrac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ is not part of SU(3) – it corresponds to a U(1).

So U(3) = U(1) ⊗ SU(3).

By design there is an obvious SU(2) subgroup provided $\lambda_{1,2,3} \leftrightarrow \sigma_{1,2,3}$.
While $\sigma_{1,2}$ have a role in forming raising and lowering operators, so will the pairs $\lambda_{1,2}$, $\lambda_{4,5}$ and $\lambda_{6,7}$.

The matrices are chosen to satisfy $\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$

The structure constants defined by $[X_a, X_b] = i\sum_c f_{abc} X_c$ are non-trivial.

# QCD [16]

We choose base vectors:

| | | |
|---|---|---|
| Red: | r or \|r> | = (1,0,0) |
| Blue: | b or \|b>. | = (0,1,0) |
| Green: | g or \|g>. | = (0,0,1) |

By design the SM has an exact $SU(3)_{colour}$ local symmetry, with a corresponding gauge invariance and the associated 8 gauge bosons – gluons.

The gauge-invariant kinetic term which can be included in the Lagrangian for the gauge fields is $L_{gauge} \sim F_{\mu\nu}F_{\mu\nu}$ where $F_{\mu\nu}$ is derived from the commutator of the covariant derivatives $D_\mu = \partial_\mu - iX^a G^a{}_\mu$:

$$F_{\mu\nu} \sim [D_\mu, D_\nu] \sim (\partial_\mu G^a{}_\nu - \partial_\nu G^a{}_\mu + f_{abc} G^b{}_\mu G^c{}_\nu) X^a$$

(For $U(1)_{EM}$, $F_{\mu\nu}$ is the field tensor, corresponding to the E and B fields.)

For a non-Abelian theory like $SU(3)_{colour}$, the structure constants are non-vanishing and there are terms in $L_{gauge}$ which correspond to triple and quartic gauge couplings, i.e. the gluons couple to themselves.

Due to a conspiracy of the QCD couplings (arising from the SU(3) properties), the energy involved in separating two coloured charges is infinite.
Therefore, free observable particles must be “colourless”, corresponding to SU(3) singlets.

(The fact that red + blue + green light appears to make white light is purely a feature of the physiology of the human eye and the fact that the cones are sensitive to red, blue and green light.)

# Colour of Hadron States [16.1]

For the description of the **baryon** colour wavefunction, we would like to construct invariant states which are "colourless".

It is tempting to consider $|r>|b>|g>$, however this is not colourless:

For example under the transformation $U = \exp(i\alpha\lambda_1) \approx 1 + i\alpha\lambda_1$

$$|r>|b>|g> \to (|r> + i\alpha|b>)(|b> + i\alpha|r>)|g> \neq |r>|b>|g>$$

We construct a (tensor) state from a linear combination:

$$\psi = \sum_{ijk} c_{ijk} \, |i>|j>|k> \quad \text{where} \quad i,j,k \text{ are taken from } \{r,b,g\}$$

Under a unitary transformation, U, $\psi \to \psi' = U\psi$

$$\psi' = \sum_{ijk} c_{ijk} U|i> U|j> U|k>$$

expanding

$$U|i> = \sum_p |p><p|U|i> = \sum_p U_{pi} \, |p>$$

$$\Rightarrow \psi' = \sum_{ijk} \sum_{pqr} c_{ijk} U_{pi} U_{qj} U_{rk} \, |p>|q>|r>$$

If we chose $c_{ijk} = \varepsilon_{ijk}$, then

$$\psi' = \sum_{ijk} \sum_{pqr} \varepsilon_{ijk} U_{pi} U_{qj} U_{rk} \, |p>|q>|r> = \sum_{pqr} \varepsilon_{pqr} \det(U) |p>|q>|r> = \sum_{pqr} \varepsilon_{pqr} \, |p>|q>|r> = \psi$$

So the colour description of a baryon is:

$$|r>|b>|g> + |g>|r>|b> + |b>|g>|r> - |r>|g>|b> - |b>|r>|g> - |g>|b>|r>$$

The conjugate state transforms: $\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}U^H$, so $<\bar{r}| \rightarrow <\bar{r}'| = <\bar{r}|U^H$

For the meson colour wavefunction, it is tempting to consider something like $|r><\bar{r}|$, however, just as before, this is not colourless:
For example under the transformation $U = \exp(i\alpha\lambda_1) \approx 1 + i\alpha\lambda_1$

$$|r><\bar{r}| \rightarrow (|r> + i\alpha|b>)(<\bar{r}| - i\alpha<\bar{b}|) \approx |r><\bar{r}| - i\alpha|r><\bar{b}| + i\alpha|b><\bar{r}| \neq |r><\bar{r}|$$

We construct a (tensor) state from a linear combination:

$\psi = \sum_{ij} c_{ij} |i><\bar{j}|$ where i,j are taken from {r,b,g}

Under a unitary transformation, U, $\psi \rightarrow \psi' = U\psi$

$$\psi' = \sum_{ij} c_{ij} U|i><\bar{j}|U^H$$

expanding

$$\psi' = \sum_{ij}\sum_{pq} c_{ij} |p><p|U|i><\bar{j}|U^H|\bar{q}><\bar{q}|$$

If we chose $c_{ij} = \delta_{ij}$, then

$$\psi' = \sum_{ij}\sum_{pq} \delta_{ij} |p><p|U|i><\bar{j}|U^H|\bar{q}><\bar{q}| = \sum_i\sum_{pq} |p><p|U|i><\bar{i}|U^H|\bar{q}><\bar{q}|$$

$$= \sum_{pq} |p><p|UU^H|\bar{q}><\bar{q}| = \sum_{pq} |p>\delta_{pq}<\bar{q}| = \sum_p |p><\bar{p}| = \psi$$

So the colour description of a meson is:

$$|r><\bar{r}| + |b><\bar{b}| + |g><\bar{g}|$$

# Gluons

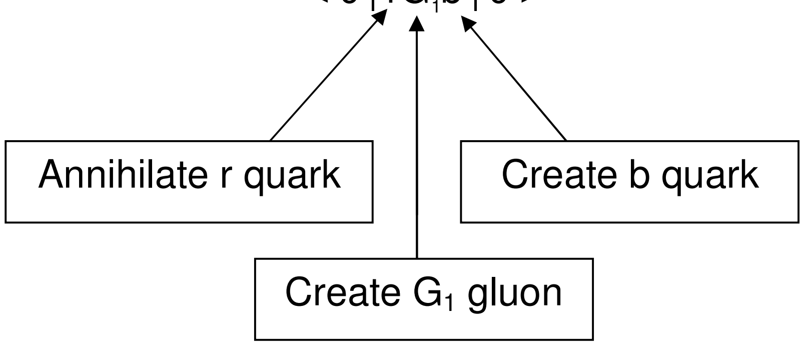
Gluons are required to ensure the invariance of the Lagrangian $L \sim \overline{\psi} D\!\!\!/ \psi$ where $D \sim \partial - i\tfrac{1}{2}\lambda_i G_i$.
This gives terms in the Lagrangian like

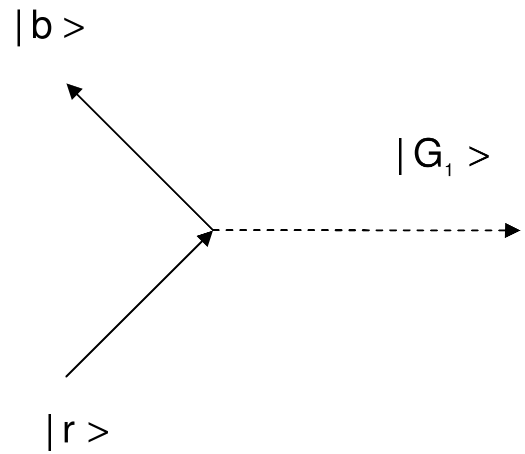
$$\overline{\psi}\lambda_1 G_1 \psi = \begin{pmatrix} \overline{r} & \overline{b} & \overline{g} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} G_1 \begin{pmatrix} r \\ b \\ g \end{pmatrix} = \overline{r} G_1 b + \overline{b} G_1 r$$

where b really corresponds to a creation operator for a blue state |b>.

We interpret these labels as operators which can operate on the vacuum.
Looking at the first term:

$$< 0 \,|\, \overline{r} G_1 b \,|\, 0 >$$




In this example, we deduce $G_1 \sim r\overline{b}$ as far as the colour quantum numbers are concerned.
So we have $G_1 \sim \frac{1}{\sqrt{2}}(r\overline{b} + b\overline{r})$

We can identify 8 coloured gluons associated with $SU(3)_{QCD}$:

$$G_1 \sim \tfrac{1}{\sqrt{2}}(r\overline{b} + b\overline{r}) \qquad G_2 \sim \tfrac{i}{\sqrt{2}}(r\overline{b} - b\overline{r}) \qquad G_3 \sim \tfrac{1}{\sqrt{2}}(r\overline{r} - b\overline{b})$$

$$G_4 \sim \tfrac{1}{\sqrt{2}}(b\overline{g} + g\overline{b}) \qquad G_5 \sim \tfrac{i}{\sqrt{2}}(b\overline{g} - g\overline{b})$$

$$G_6 \sim \tfrac{1}{\sqrt{2}}(g\overline{r} + r\overline{g}) \qquad G_7 \sim \tfrac{i}{\sqrt{2}}(g\overline{r} - r\overline{g}) \qquad G_8 \sim \tfrac{1}{\sqrt{6}}(r\overline{r} + b\overline{b} - 2g\overline{g})$$

– the coefficients from the Langrangian are absorbed into the description of the gluon wave-functions.

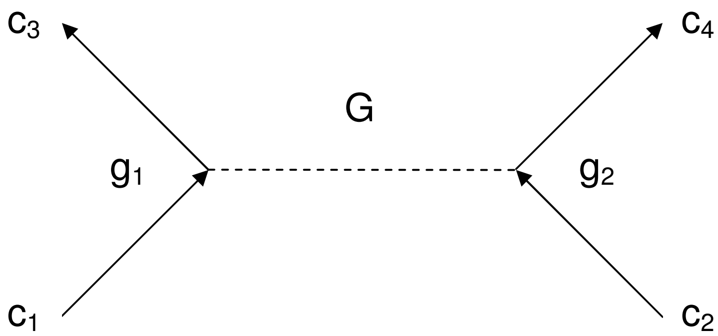
The singlet $\tfrac{1}{\sqrt{3}}(r\overline{r} + b\overline{b} + g\overline{g})$,corresponding to a U(1), is not “observed”.

In analogy with $W^{\pm} = \frac{1}{\sqrt{2}}(W_1 \mp iW_2)$, the states can be combined to create “charged operators”:

$$\tfrac{1}{\sqrt{2}}(G_1 \mp iG_2) = r\overline{b} \quad \text{etc}$$

These represent the flow of colour “charge”, corresponding to the exchange of gluons from one quark to another.

# Colour Factors

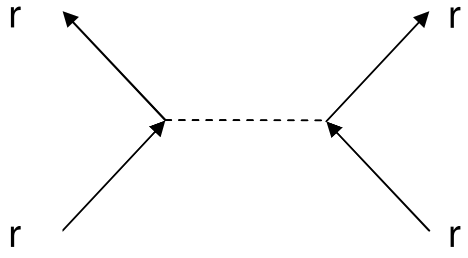



The **scattering amplitude** for the above process $\propto g_1 g_2$.

The coupling strengths at each vertex are found from the projection of the colour state $|c_i\rangle|c_f\rangle$ on to the gluon state $|G\rangle$.

According to the rules of QFT, we include a –ve sign for antiquarks.

**Similar quarks:** $qq \to qq$. E.g. $rr \to rr$




Exchange gluons:

$$G_3 \sim \tfrac{1}{\sqrt{2}}(r\bar{r} - b\bar{b})$$
$$G_8 \sim \tfrac{1}{\sqrt{6}}(r\bar{r} + b\bar{b} - 2g\bar{g})$$

Amplitude for $rr \to rr$:

$A(rr \to rr) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$ via $G_3$ and $A(rr \to rr) = \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}}$ via $G_8$.

So

$$A(rr \to rr) = \tfrac{1}{\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{6}} \cdot \tfrac{1}{\sqrt{6}} = \tfrac{2}{3}$$

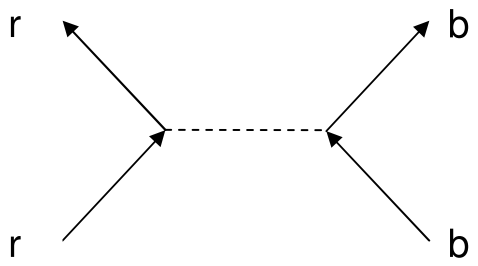
Likewise, if we consider $bb \to bb$ and $gg \to gg$, we find:

$$A(bb \to bb) = \tfrac{-1}{\sqrt{2}} \cdot \tfrac{-1}{\sqrt{2}} + \tfrac{1}{\sqrt{6}} \cdot \tfrac{1}{\sqrt{6}} = \tfrac{2}{3}$$
$$A(gg \to gg) = \tfrac{-2}{\sqrt{6}} \cdot \tfrac{-2}{\sqrt{6}} = \tfrac{2}{3}$$

They are all the same – there is invariance to the colour of the quark … would expect this if there is to be colour symmetry.

$$A(qq \to qq) = \tfrac{2}{3}$$

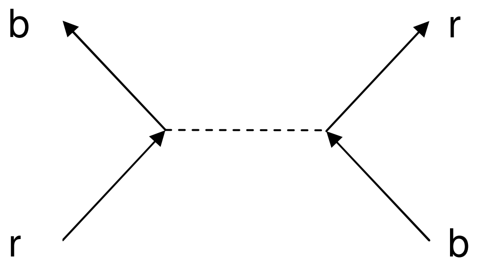
**Different quarks:** $qq' \rightarrow qq'$. E.g. $rb \rightarrow rb$




Exchange gluons:

$$G_3 \sim \tfrac{1}{\sqrt{2}}(r\bar{r} - b\bar{b})$$
$$G_8 \sim \tfrac{1}{\sqrt{6}}(r\bar{r} + b\bar{b} - 2g\bar{g})$$

$\pm$ ??



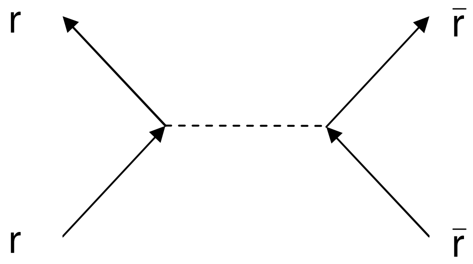

$r\bar{b}$

$$A(rb \rightarrow rb) = \tfrac{1}{\sqrt{2}} \cdot \tfrac{-1}{\sqrt{2}} + \tfrac{1}{\sqrt{6}} \cdot \tfrac{1}{\sqrt{6}} = \tfrac{-1}{3}$$
$$A(rb \rightarrow br) = 1 \cdot 1 = 1$$

But do we add or subtract the amplitudes ?
This depends on the symmetry of the quark system.

$$A(qq' \rightarrow qq') = \tfrac{-1}{3} \pm 1$$

**Quark & Antiquark:** $q\bar{q} \to q\bar{q}$. E.g. $r\bar{r} \to r\bar{r}$




Exchange gluons:

$G_3 \sim \frac{1}{\sqrt{2}}(r\bar{r} - b\bar{b})$

$G_8 \sim \frac{1}{\sqrt{6}}(r\bar{r} + b\bar{b} - 2g\bar{g})$

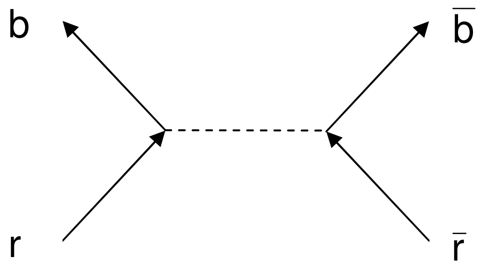
So

$$A(r\bar{r} \to r\bar{r}) = \frac{1}{\sqrt{2}} \cdot \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{6}} \cdot \frac{-1}{\sqrt{6}} = \frac{-2}{3}$$

$$A(q\bar{q} \to q\bar{q}) = \frac{-2}{3}$$

Note: we have ignored the s-channel scattering:

**Quark & Antiquark:** $q\bar{q} \rightarrow q'\bar{q}'$. E.g. $r\bar{r} \rightarrow b\bar{b}$




Exchange gluons:

$r\bar{b}$

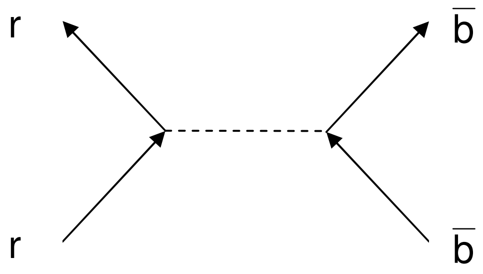
So

$$A(r\bar{r} \rightarrow b\bar{b}) = 1 \cdot -1 = -1$$

$$A(q\bar{q} \rightarrow q'\bar{q}') = -1$$

Again the s-channel scattering has been ignored.

## **Quark & Antiquark:** $q\bar{q}' \to q\bar{q}'$. E.g. $r\bar{b} \to r\bar{b}$




Exchange gluons:

$$G_3 \sim \tfrac{1}{\sqrt{2}}(r\bar{r} - b\bar{b})$$
$$G_8 \sim \tfrac{1}{\sqrt{6}}(r\bar{r} + b\bar{b} - 2g\bar{g})$$

So

$$A(r\bar{b} \to r\bar{b}) = \tfrac{1}{\sqrt{2}} \cdot \tfrac{+1}{\sqrt{2}} + \tfrac{1}{\sqrt{6}} \cdot \tfrac{-1}{\sqrt{6}} = \tfrac{1}{3}$$

$$A(q\bar{q}' \to q\bar{q}') = \tfrac{1}{3}$$

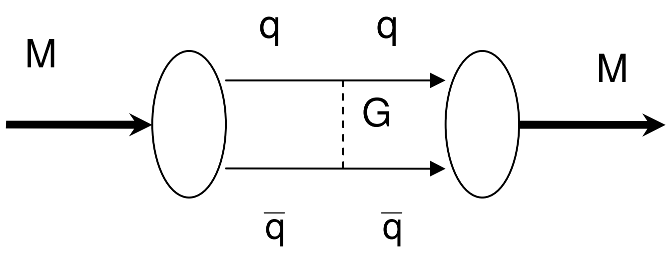
Summary:

$$A(qq \to qq) = \tfrac{2}{3} \qquad A(q\bar{q} \to q\bar{q}) = \tfrac{-2}{3} \qquad A(q\bar{q} \to q'\bar{q}') = -1$$

$$A(qq' \to qq') = \tfrac{-1}{3} \pm 1 \qquad A(q\bar{q}' \to q\bar{q}') = \tfrac{1}{3}$$

# Example using Colour Factors

Consider interactions in a $q\bar{q}$ (e.g. a meson):




Meson: $|M> = \frac{1}{\sqrt{3}}|r\bar{r} + b\bar{b} + g\bar{g}>$

Amplitude is $<M|\sum G|M>$ where the sum is over all gluons.

Using previous results:

$$<r\bar{r} + b\bar{b} + g\bar{g}|\sum G|r\bar{r}> = \frac{-2}{3} + -1 + -1 = \frac{-8}{3}$$

Need two lots of $\frac{1}{\sqrt{3}}$ and also consider $|b\bar{b}>$ and $|g\bar{g}>$ (same as $|r\bar{r}>$ by symmetry).

So amplitude $\sim 3 \times (\frac{1}{\sqrt{3}})^2 \times \frac{-8}{3} = \frac{-8}{3}$

# Weights [7.2]

The commuting generators in SU(3)(Cartan Subalgebra) are:

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \lambda_8 = \tfrac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

This implies there are two simultaneously observable quantum numbers, along with $I^2$.

With an eye to hadrons, rather than QCD, we define

Isospin $\quad I_3 = \tfrac{1}{2}\lambda_3$

Hypercharge $\quad Y = \tfrac{1}{\sqrt{3}}\lambda_8$

The weights are readily identified from the diagonal matrices:

$I_3 = \tfrac{+1}{2}, \tfrac{-1}{2}, 0$ and $Y = \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{-2}{3}$

Just as in SU(2), we defined raising and lowering operators which move between the different weight vectors (in SU(2), points on line), so we can define raising and lowering operators which move between the different weight vectors, which in SU(3), will be points in the plane.

$$I_\pm = \tfrac{1}{2}(\lambda_1 \pm i\lambda_2) \quad U_\pm = \tfrac{1}{2}(\lambda_6 \pm i\lambda_7) \quad V_\pm = \tfrac{1}{2}(\lambda_4 \pm i\lambda_5)$$

These are not all independent.

# Quark Flavour SU(3) [11]

The quarks (u, d, s) are all light (compared to hadron masses) and their interactions are dominated by the flavour-independent colour force.

We choose base vectors:

| | | |
|---|---|---|
| Up: | u | = (1,0,0) |
| Down: | d | = (0,1,0) |
| Strange: | s | = (0,0,1) |

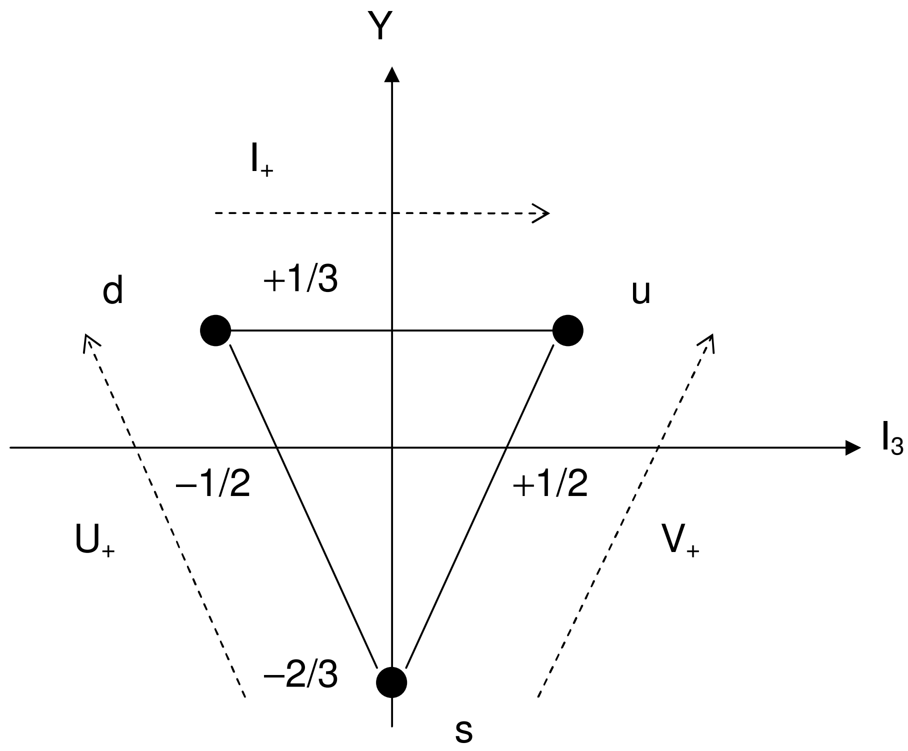
The weights are:

| | $I_3$ | Y |
|---|---|---|
| u | +1/2 | +1/3 |
| d | –1/2 | +1/3 |
| s | 0 | –2/3 |

Just as $I_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ raises d = (0,1,0) to u = (1,0,0)

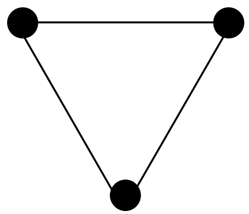
So $U_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ raises s = (0,0,1) to d = (0,1,0)

The **weight diagram** looks like:

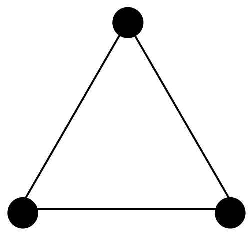



We simplify this to the fundamental representation for the quarks (**3**) and the corresponding weight diagram for the antiquarks ($\bar{\mathbf{3}}$) (negated quantum numbers):

**3** =

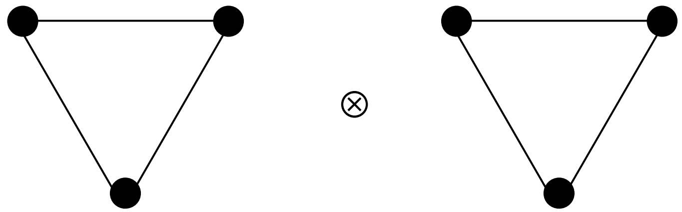


$\bar{\mathbf{3}}$ =

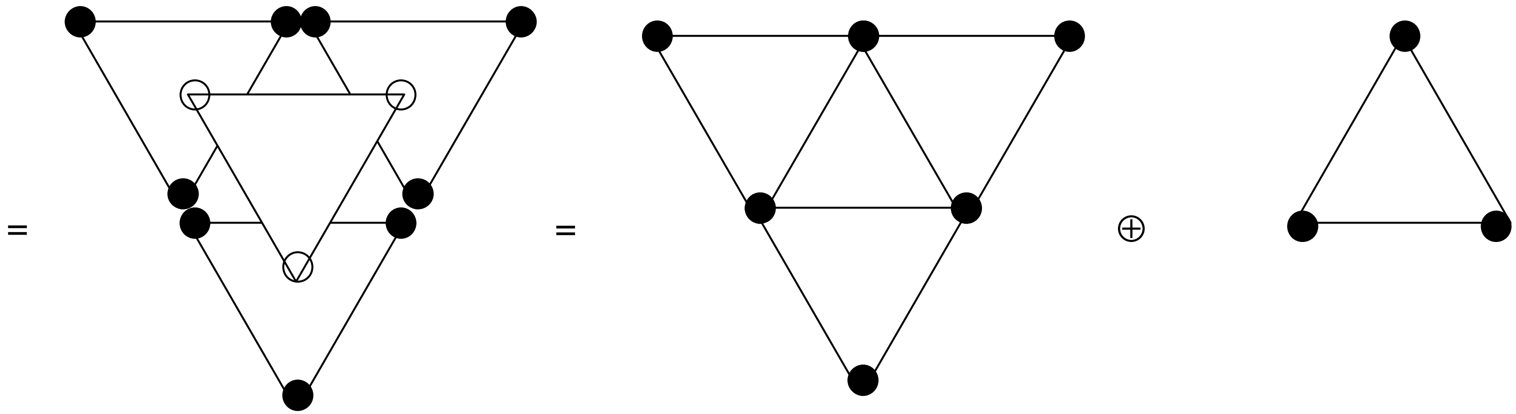


To identify the states and multiplets for combining quarks and antiquarks, we “add” one diagram to the vertices of the previous one:

E.g. combining 2 quarks:

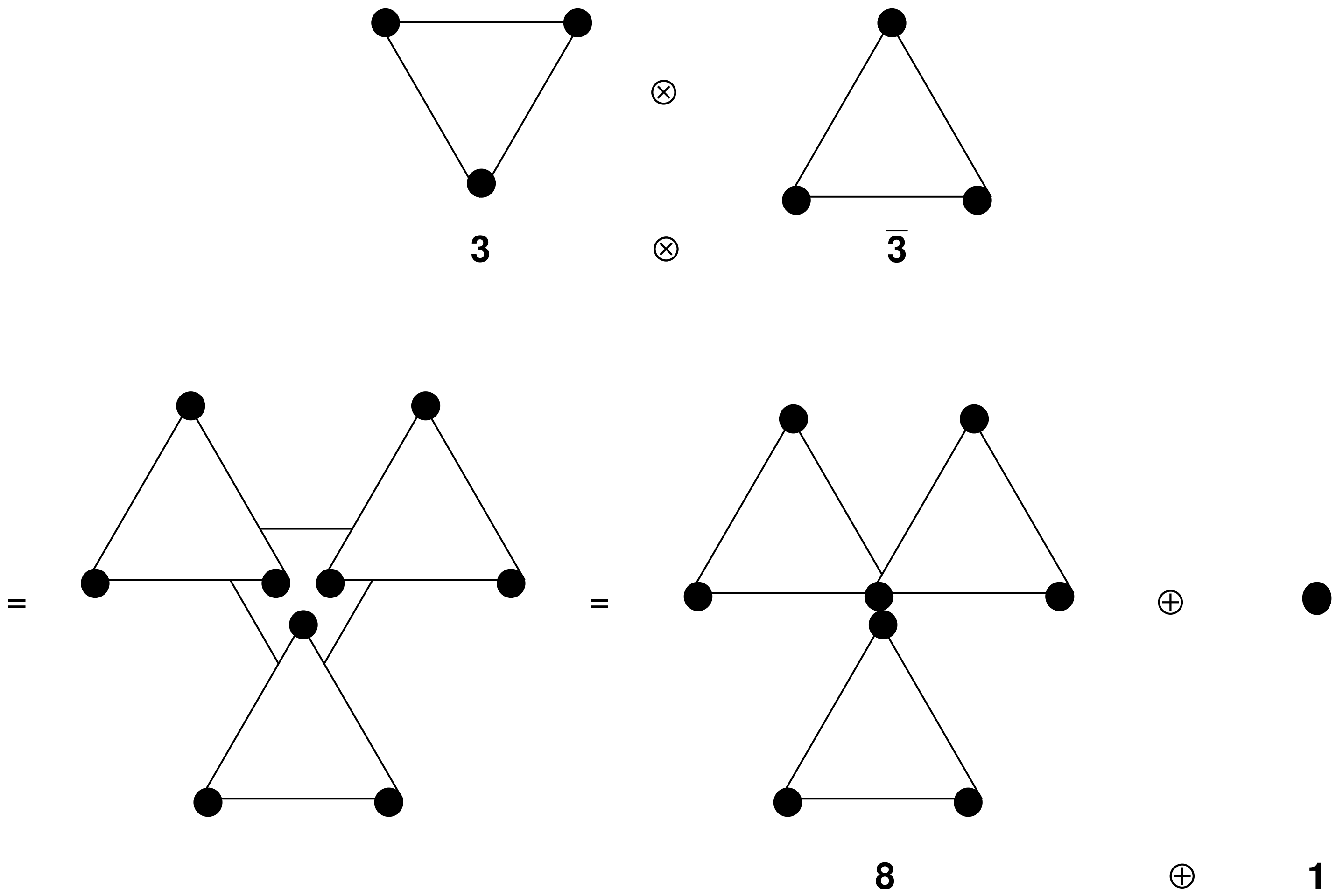


$\mathbf{3} \otimes \mathbf{3}$



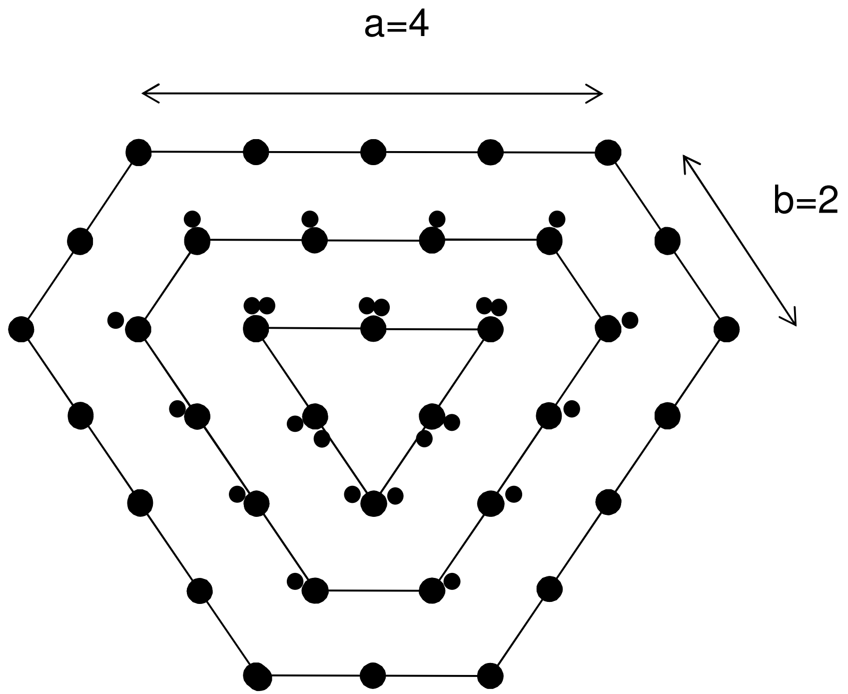
$= \mathbf{6} \oplus \overline{\mathbf{3}}$

E.g. combining quark and antiquark:



These diagrams show the weights (quantum numbers), but how does one identify the multiplets associated with the symmetry ?

General multiplet with 6-sides and 3-fold symmetry:

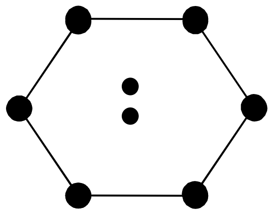



Rules:

- On outer ring, only 1 state.
- On each subsequent inner ring, add an extra state at each node …
- Until a triangle is obtained.
- After this, all triangles have the same number of states.

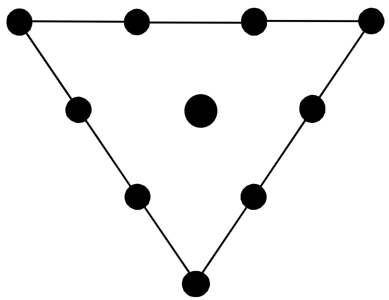
The multiplet multiplicity is given by ½ (a+1)(b+1)(a+b+2).

E.g.



has a = b = 1, so multiplicity is $\frac{1}{2} \times 2 \times 2 \times 4 = 8$ – correct

Having said the above, the most complex multiplicity we ever get to worry about is for 3 quarks or antiquarks.
So in $SU(3)_{flavour}$, the largest multiplicity corresponds to the 10:



a = 3, b = 0, so multiplicity is $\frac{1}{2} \times 4 \times 1 \times 5 = 10$ – correct

# Multiparticle States [12]

Eigenstates describing the combination of individual identical particles tend to have a well-defined exchange symmetry.
The symmetry operators will not change the exchange symmetry – the symmetry operators commute with the exchange operator:
If $U = \exp(i\alpha X_{12})$ where $X_{12} = X_1 + X_2$ and $X_i$ operates on particle i, and $P_{12}$ is the exchange operator such that particles 1 and 2 swap quantum numbers,

$$P_{12}\, U = P_{12} \exp(i\alpha\,(X_1 + X_2)) = \exp(i\alpha\,(X_2 + X_1))\, P_{12} = \exp(i\alpha\,(X_1 + X_2))\, P_{12} = U\, P_{12}$$

This means that the symmetry operations will not modify the exchange symmetry of a state.
Since the multiplets consist of those states related to each other by unitary transformations (in particular, the raising & lowering operators) from each other, they will all have the same symmetry.

We demonstrated explicitly in the last lecture that the I=0 and I=1 states transformed within their multiplets.

These multiplets are the irreducible representations.

# Construction States – Young Tableaux

We consider a 2-particle wavefunction: $\psi_{12}$ – where the index 1 (2) describes the quantum numbers of particle #1 (#2).

The exchange (or permutation) operator can be used to generate states of explicit symmetry:

$S_{12} = 1 + P_{12}$ symmetrising operator

$A_{12} = 1 - P_{12}$ antisymmetrising operator

$P_{12} P_{12} = 1$, so $P_{12} S_{12} = P_{12} + 1 = S_{12}$ and $P_{12} A_{12} = P_{12} - 1 = -A_{12}$

So

$P_{12} (S_{12}\psi_{12}) = + (S_{12}\psi_{12})$ and $P_{12} (A_{12}\psi_{12}) = - (A_{12}\psi_{12})$

Starting from a particular multiparticle state, we can apply $S_{ij}$ and $A_{ij}$ repeatedly to build up states which under exchange are:

- Symmetric with respect to all particles
- Antisymmetric with respect to all particles
- Mixed symmetry, which may be symmetric (antisymmetric) with respect to particular particles.

Rather than "ψ" on to which we hang quantum number labels, we use:

☐ to denote a particle – of which there are $N_p$.

i to denote the state of a particle – of which there are $N_n$.

Complete states for $N_p$ particles are denoted by an arrangement of $N_p$ ☐'s, each with its own quantum number label:

E.g.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 3 | 3 | 3 |
| 4 | 5 | 6 | | | | |
| 9 | | | | | | |
| 10 | | | | | | |

The numbers are the single-particle quantum numbers (could be labelled a, b, c … or α, β, γ …) and must not exceed $N_n$.

The states are symmetrised with respect to all the particles (☐'s) in a given *row* and
The states are antisymmetrised with respect to all the particles (☐'s) in a given *column*.

The rules for constructing the Tableaux are:

1. A row must not be longer than the one above it.
2. The numbers (quantum number labels) when viewed in reading order through the table must not decrease.
3. Going down vertically in a given column, the numbers must increase.

The rules ensure that don't

- double count ($2^{nd}$ rule)
- antisymmetrise wrt same single particle state, causing a vanishing combination

# Example for SU(3)

Consider a two particle state: $N_p = 2$ and we have two □'s.
In SU(3), there are 3 labels, e.g. (u,d,s) – we will call them generically (1,2,3) – $N_n = 3$.

We can construct 6 symmetric states:

| $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$ | $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ | $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}$ | $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}$ | $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}$ | $\begin{array}{|c|c|}\hline 3 & 3 \\ \hline\end{array}$ |
|---|---|---|---|---|---|
| uu | $\frac{1}{\sqrt{2}}(ud+du)$ | $\frac{1}{\sqrt{2}}(us+su)$ | dd | $\frac{1}{\sqrt{2}}(ds+sd)$ | ss |

and 3 antisymmetric states:

| $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$ | $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}$ | $\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$ |
|---|---|---|
| $\frac{1}{\sqrt{2}}(ud-du)$ | $\frac{1}{\sqrt{2}}(us-su)$ | $\frac{1}{\sqrt{2}}(ds-sd)$ |

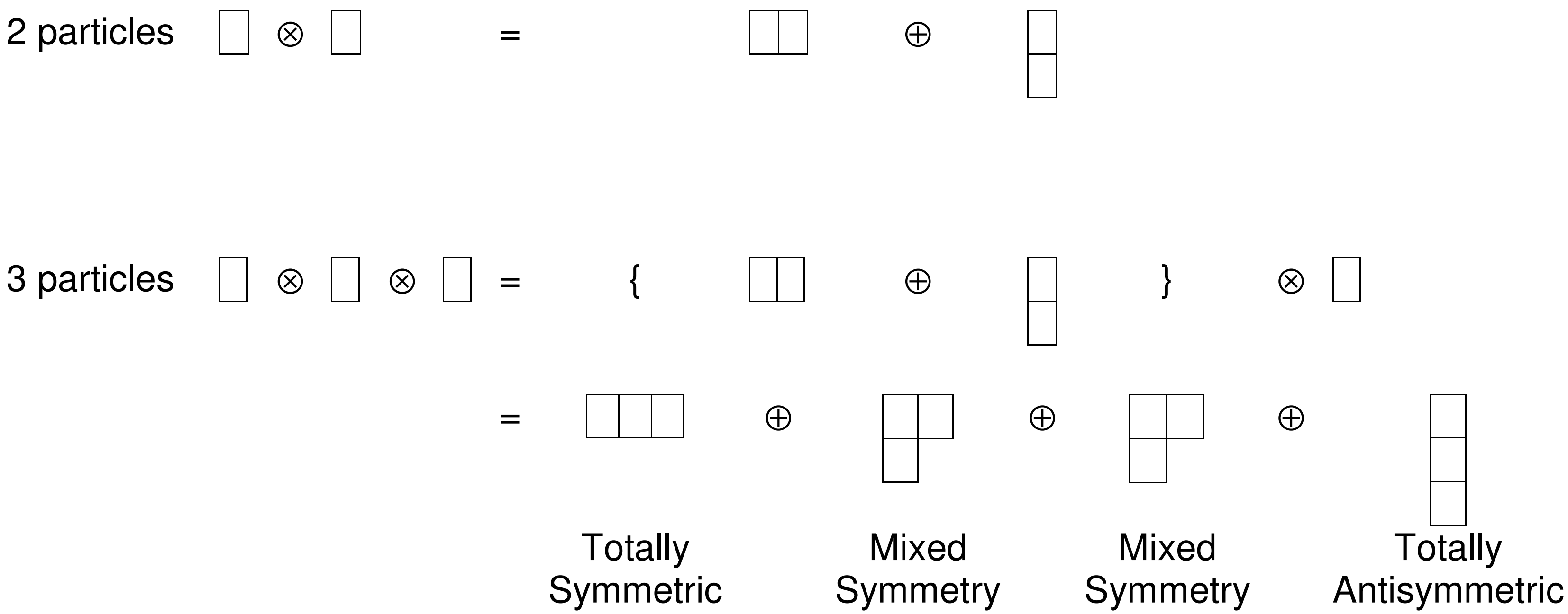
We have already seen $uu$, $\frac{1}{\sqrt{2}}(ud+du)$, $dd$, $\frac{1}{\sqrt{2}}(ud-du)$ when considering the SU(2) subgroup.

The shape of the Tableaux corresponds to the multiplets of the representations.
Having motivated the Young Tableaux, we will drop the state labels.

# Combining Multiplets

All that is really needed are fairly simple combinations; hence, crudely speaking, it is sufficient to simply combine diagrams in a manner consistent with the Rules. (For more details, see [12.2].)

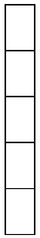
Some examples:



Recall that a totally antisymmetric singlet could be generated by $\varepsilon_{ijk\ldots}\, q_i\, q_j\, q_k \ldots$

To the extent that this can be considered to be the "vacuum" state, then removing one quark gives rise to a description of the **conjugate** or **antiparticle state**: $\varepsilon_{ijk\ldots}\, q_i\, q_j \ldots$

So in SU(n), the corresponding Young Tableaux can be represented by a column of $N_n-1$ □'s:

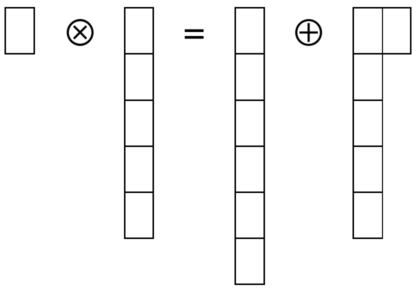


$N_n-1$ □'s denoted $\bar{\mathbf{n}}$

Note in SU(2), the conjugate is $\bar{\mathbf{2}}$ = □ … which is the same as the quark state **2** = □.

We know this because we showed that the conjugate state $\bar{\mathbf{2}} = \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$ transforms like $\mathbf{2} \equiv \begin{pmatrix} u \\ d \end{pmatrix}$.

Combining a quark and an antiquark under SU(n):



where the first multiplet is a singlet in SU(n).

# Calculating Multiplicities [13.3]

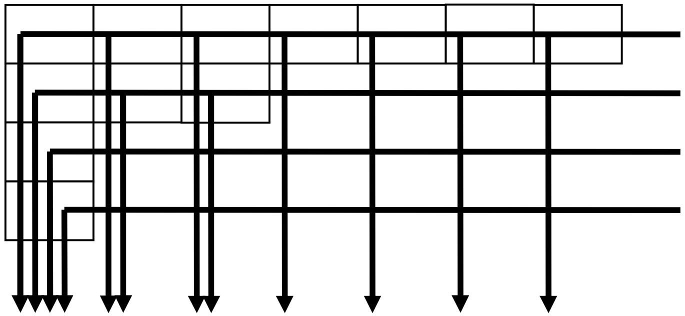
The beauty of the Young Tableaux is that they help us identify multiplets, understand their symmetry and evaluate their multiplicities.

The multiplicity is a ratio:
For *numerator* in SU(n), insert numbers:

| | | | | | | |
|---|---|---|---|---|---|---|
| n | n+1 | n+2 | n+3 | n+4 | n+5 | n+6 |
| n–1 | n | n+1 | | | | |
| n–2 | | | | | | |
| n–3 | | | | | | |

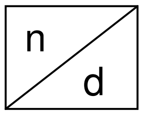
and take the product.

For *denominator* count the length of the **hooks**



| | | | | | | |
|---|---|---|---|---|---|---|
| 10 | 7 | 6 | 4 | 3 | 2 | 1 |
| 5 | 2 | 1 | | | | |
| 2 | | | | | | |
| 1 | | | | | | |

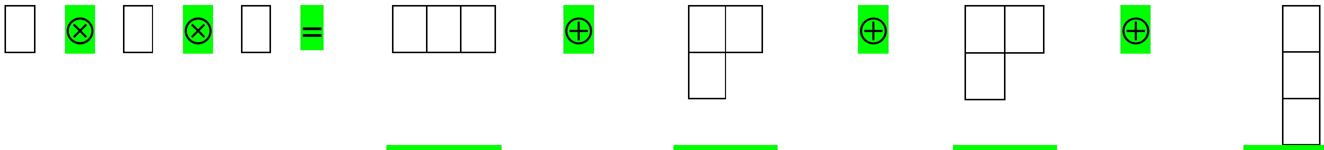
and take the product.

Can include both numbers in a cell with a diagonal: 

<table>
<tr><th></th><th></th><th>numerator</th><th>denominator</th><th>multiplicity</th></tr>
<tr><td>n</td><td>$\begin{array}{|c|}\hline n\\ \hline\end{array}$</td><td>n</td><td>1</td><td>n</td></tr>
<tr><td>$\bar{\mathbf{n}}$</td><td>$\begin{array}{|c|}\hline n\\ \hline n-1\\ \hline \dots\\ \hline 2\\ \hline\end{array}$</td><td>n(n–1) … 2</td><td>(n–1) … 1</td><td>n</td></tr>
<tr><td>1</td><td>$\begin{array}{|c|}\hline n\\ \hline n-1\\ \hline \dots\\ \hline 1\\ \hline\end{array}$</td><td>n!</td><td>n!</td><td>1</td></tr>
<tr><td></td><td>$\begin{array}{|c|c|}\hline n & n+1\\ \hline n-1 & \\ \cline{1-1}\end{array}$</td><td>n(n+1)(n–1)</td><td>3·1·1</td><td>$\dfrac{(n-1)n(n+1)}{3}$</td></tr>
</table>

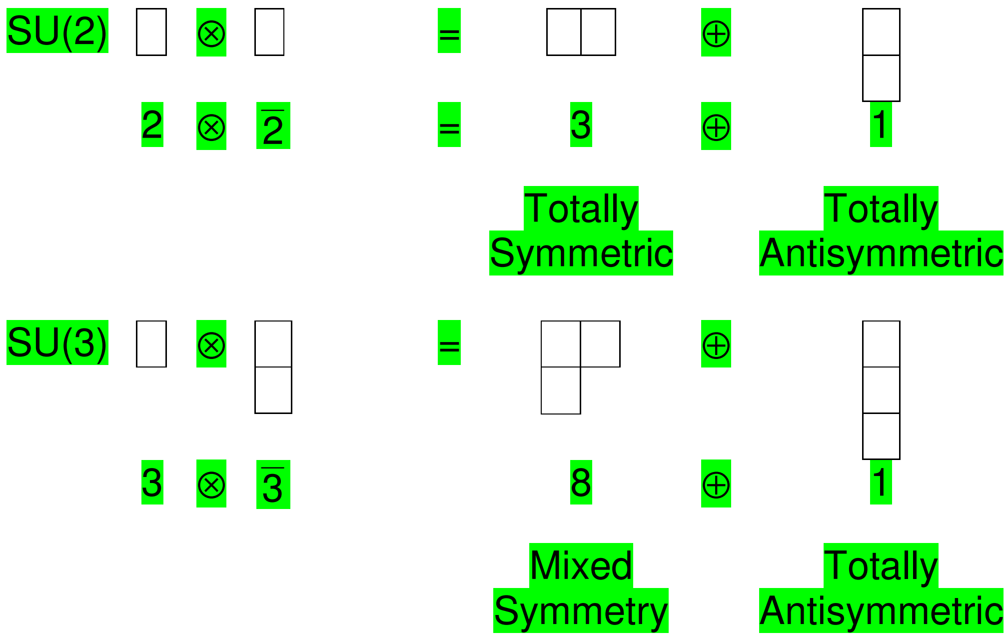
E.g. SU(2) → 2
SU(3) → 8

**3 Quarks**:

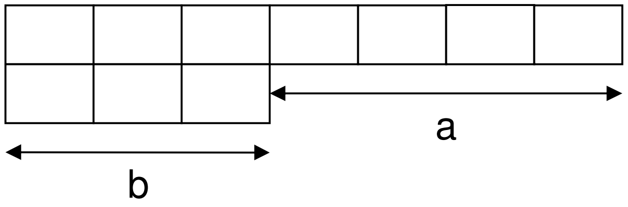


| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | Totally Symmetric | | Mixed Symmetry | | Mixed Symmetry | | Totally Antisymmetric |
| SU(2) | 2 | $\otimes$ | 2 | $\otimes$ | 2 | = | 4 | $\oplus$ | 2 | $\oplus$ | 2 | $\oplus$ | 0 |
| SU(3) | 3 | $\otimes$ | 3 | $\otimes$ | 3 | = | 10 | $\oplus$ | 8 | $\oplus$ | 8 | $\oplus$ | 1 |
| SU(6) | 6 | $\otimes$ | 6 | $\otimes$ | 6 | = | 56 | $\oplus$ | 70 | $\oplus$ | 70 | $\oplus$ | 20 |

**Quark and Antiquark:**



The most general Tableaux for SU(2) has the form




There cannot be a third row, since there are only two labels and in a given column, the labels cannot be the same.
This state corresponds to $N_p = a + 2b$ particles.

If we label the configuration with the labels for SU(2) of Quark Flavour (u,d), the first such state might be:

| | | | | | | |
|---|---|---|---|---|---|---|
| u | u | u | u | u | u | u |
| d | d | d | | | | |

Since the corresponding wavefunction must be antisymmetrised with respect to the labels in the columns, these will consist of pairs ~ (ud–du). These correspond to states I=0, $I_3$=0.
Therefore the Tableaux corresponds to a state with I = ½ a, $I_3$ = ½ a.

The next state to construct would be:

| | | | | | | |
|---|---|---|---|---|---|---|
| u | u | u | u | u | u | d |
| d | d | d | | | | |

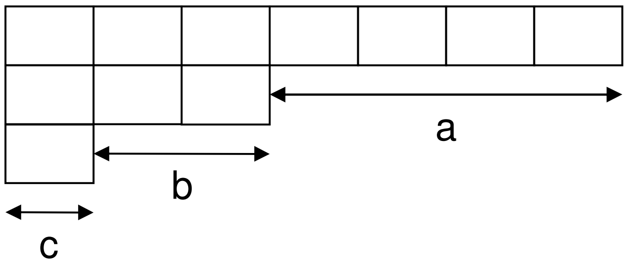
Effectively, this can be obtained be obtained by applying a lowering operator.

The last state in the series will be:

| | | | | | | |
|---|---|---|---|---|---|---|
| u | u | u | d | d | d | d |
| d | d | d | | | | |

a total of (a+1) states corresponding to I = ½ a.

The most general Tableaux for SU(3) has the form




This has a multiplicity ½ (a+1)(b+1)(a+b+2).

<u>Homework</u>

Verify the multiplicities for the general SU(2) and SU(3) Tableaux using the numerator/denominator method.